Student Workbook

Solving Three-Variable Linear Systems

Name: ______________________ Date: ____________ Class: ________

ESSENTIAL QUESTION

How does back-substitution after elimination allow you to solve a 3×3 linear system, and what does it mean geometrically when a 3-variable system has no solution or infinitely many solutions?

Lesson Overview

A 3×3 linear system consists of three equations in three unknowns (x, y, z). Each equation represents a plane in 3D space. Three planes intersect at exactly one point for a unique solution.

Eliminate → Reduce to 2×2 → Solve → Back-substitute

Unique	Three planes intersect at exactly one point.
No Sol.	Planes are parallel or form a "triangular prism" with no common point.
Infinite	Planes intersect along a line, or all three are the same plane.
Strategy	Choose a variable to eliminate; use pairs of equations to eliminate it; reduce to a 2×2 system; solve; back-substitute.
Row Ops	Swap two equations. Multiply an equation by a nonzero constant. Add a multiple of one equation to another.

Worked Examples

EXAMPLE 1

Solve: $x + y + z = 6$, $2x - y + z = 3$, $x + 2y - z = 2$

- Label (1) $x+y+z=6$ (2) $2x-y+z=3$ (3) $x+2y-z=2$
- (1)+(3): $2x+3y=8$ (A). (2)+(3): $3x+y=5$ (C)
- From (C): $y=5-3x$. Sub into (A): $2x+3(5-3x)=8 \to -7x=-7 \to x=1$; $y=2$; from (1): $z=3$

Answer: (1, 2, 3)

EXAMPLE 2

Solve: $2x + y - z = 8$, $-3x - y + 2z = -11$, $-2x + y + 2z = -3$

- (1)+(2): $-x+z=-3$ (A). (1)+(3): $2y+z=5$ (B). (2)+(3): $-5x+4z=-14$ (C)
- From (A): $z=x-3$. Sub into (C): $-5x+4(x-3)=-14 \to -x-12=-14 \to x=2$
- $z=2-3=-1$. From (B): $2y+(-1)=5 \to y=3$

Answer: (2, 3, −1)

EXAMPLE 3

Identify the system type: $x + 2y - z = 4$, $2x + 4y - 2z = 8$, $-x - 2y + z = -4$

- Eq(2) = 2 × Eq(1): $2(x+2y-z)=2(4)=8$ ✓
- Eq(3) = −1 × Eq(1): $-(x+2y-z)=-4$ ✓
- All three equations represent the same plane

Answer: Dependent — infinitely many solutions

EXAMPLE 4

Identify the system type: $x + y + z = 3$, $2x + 2y + 2z = 7$, $x - y + z = 1$

- 2 × Eq(1) gives $2x+2y+2z=6$
- Eq(2) states $2x+2y+2z=7$ — contradiction

Answer: Inconsistent — no solution

EXAMPLE 5

Curve fitting: Find a, b, c so $y = ax^2 + bx + c$ passes through (1,2), (2,5), (3,10)

- Substitute each point: $a+b+c=2$ (1); $4a+2b+c=5$ (2); $9a+3b+c=10$ (3)
- (2)−(1): $3a+b=3$ (A). (3)−(2): $5a+b=5$ (B)
- (B)−(A): $2a=2 \rightarrow a=1$. From (A): $b=0$. From (1): $c=1$

Answer: $y = x^2 + 1$

Guided Practice

Work through each problem below. Use the hint if you get stuck — write your final answer on the last line.

1 Solve: $x + y + z = 0$, $2x - y + 3z = -1$, $x + 2y - z = 4$.

Hint: Eliminate z by adding equations in pairs.

2 Solve: $2x + y + z = 7$, $x - y + 2z = 4$, $3x + 2y - z = 10$.

Hint: Start by eliminating y from two pairs of equations.

3 Determine the type: $x + y - z = 2$, $2x + 2y - 2z = 4$, $3x + 3y - 3z = 6$.

Hint: Compare the equations — are they multiples of each other?

4 Determine the type: $x + y + z = 5$, $2x + 2y + 2z = 9$, $x - y + z = 1$.

Hint: Check if equation 2 is consistent with equation 1.

5 A parabola $y = ax^2 + bx + c$ passes through $(-1, 6)$, $(0, 1)$, and $(2, 3)$. Find a, b, c.

Hint: Substitute each point to get three equations in a, b, c.

Key Vocabulary

3×3 System	Three linear equations in three unknowns x, y, z. Ex: $x+y+z=6$, $2x-y+z=3$, $x+2y-z=2$.
Back-Substitution	After solving for one variable, substituting its value into other equations to find the rest.
Row Operations	Swapping equations, multiplying by a nonzero constant, or adding a multiple of one to another — all preserve the solution set.
Triangular Form	A system where eq 1 has 3 variables, eq 2 has 2, and eq 3 has 1 — making back-substitution straightforward.
Dependent (3D)	The three planes intersect along a line or are all the same plane — infinitely many solutions.
Inconsistent (3D)	The planes have no common intersection point — no solution exists.

Quick Check Quiz

Circle the letter of the correct answer for each question.

1 **The solution to a 3×3 system represents geometrically:** Multiple Choice

A A line in 3D
B A point in 3D
C A plane in 3D
D A curve in 3D

2 **In a 3×3 system, if elimination gives 0 = 0, the system is:** Multiple Choice

A Inconsistent
B Consistent and independent
C Consistent and dependent
D Underdetermined

3 **Which is a valid row operation?** Multiple Choice

- **A** Multiply one equation by 0
- **B** Add a multiple of one equation to another
- **C** Divide by a variable
- **D** Square both sides

4 **Solve: $x + y = 3$, $y + z = 5$, $x + z = 4$.** Multiple Choice

- **A** (1, 2, 3)
- **B** (2, 1, 4)
- **C** (1, 3, 2)
- **D** (3, 2, 1)

5 **Back-substitution is used after:** Multiple Choice

- **A** Graphing
- **B** Reducing to triangular form
- **C** Finding the determinant
- **D** Completing the square

Common Mistakes to Avoid

✗ Eliminating different variables in each step without a plan.

✓ Choose ONE variable to eliminate first. Use it to reduce all three equations to a 2×2 system, then solve that.

✗ Forgetting to use all three equations — only using two pairs.

✓ To eliminate one variable from a 3×3 system, you need to use it in at least two different pairs of equations.

✗ Sign errors when subtracting equations.

✓ When subtracting, distribute the negative sign to EVERY term. Consider adding the negative of an equation instead.

✗ Not checking the solution in all three original equations.

✓ A 3-variable solution must satisfy all three equations. Always verify by substituting back.

Math Tips

★ Organize your work: label each equation (1), (2), (3) and each derived equation (A), (B). This prevents confusion.

★ The goal is to reduce the 3×3 system to a 2×2 system, then to a 1×1 equation. Work systematically.

★ For curve fitting through 3 points, substitute each point into $y=ax^2+bx+c$ to get three equations in a, b, c.

★ If two equations are multiples of each other, the system is either dependent (same equation) or inconsistent (contradictory constants).

★ Always verify your solution in ALL THREE original equations — not just the last one you used.